Ore Extensions Satisfying a Polynomial Identity

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Abstract

Necessary and sufficient conditions for an Ore extension $S = R[x;\sigma,\delta]$ to be a PI ring are given in the case σ is an injective endomorphism of a semiprime ring R satisfying the ACC on annihilators. Also, for an arbitrary endomorphism τ of R, a characterization of Ore extensions $R[x;\tau]$ which are PI rings is given, provided the coefficient ring R is noetherian.

Introduction

The aim of the paper is to give necessary and sufficient conditions for an Ore extension $R[x;\sigma,\delta]$ to satisfy a polynomial identity. One of the special feature is that we do not assume that σ is an automorphism.

Clearly if $R[x;\sigma,\delta]$ satisfies a polynomial identity, then R has to be a PI ring as well. Henceforth we will always assume that R is a PI ring.

In [14], Pascaud and Valette showed that when R is semiprime and σ is an automorphism of R, then the Ore extension $R[x;\sigma]$ satisfies a polynomial identity if and only if σ is of finite order on the center of R. We shall obtain similar results even when σ is not an automorphism.

The PI property of general Ore extensions $R[x;\sigma,\delta]$ was studied by Cauchon in his thesis [4] in the case when the base ring R is simple and σ is

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an automorphism of R. In particular, Cauchon remarked that nonconstant central polynomials of $R[x;\sigma,\delta]$ appears naturally in this context.

On the other hand, the case of a noetherian base ring was considered by Damiano and Shapiro in [6]. They proved, in particular, that the Ore extension $R[x;\sigma]$ over a noetherian PI ring R satisfies a polynomial identity if and only if the automorphism σ is of finite order on the center of R/B, where B denotes the prime radical of R. This result will be generalized to arbitrary endomorphism of R in the last section.

Let us also mention that, mainly for Ore extensions coming from quantum groups, some authors are also interested in the relations between the PI degree of a ring R and the PI degree of Ore extensions over R (Cf. e.g. [3],[8]).

A somewhat related work is that of Bergen and Grzeszczuk (Cf. [1]), where a characterization of smash products $R\#U(L)$ satisfying polynomial identity is given, where R is a semiprime algebra of characteristic 0 acted by a Lie color algebra L.

In Section 1, we recall some classical results and develop tools that will play an essential role in later sections.

In Section 2, we analyse the special case when R is a prime ring. The main results being Theorem 2.7, Theorem 2.10 and Theorem 2.12. We prove, in particular, that for a prime PI ring R and an injective endomorphism σ of R, $R[x;\sigma,\delta]$ is PI if and only if there exists a nonconstant polynomial in the center of $R[x;\sigma,\delta]$ with regular leading coefficient if and only if the center $Z(R)$ of R has finite uniform dimension over $Z(R)^{\sigma,\delta} := \{z \in Z(R) \mid \sigma(z) = z,\ \delta(z) = 0\}$.

In Section 3, we extend the results from the previous section to the case of a semiprime coefficient ring R satisfying the ACC on annihilators. We first recall the results of Cauchon and Robson related to the action of an injective endomorphism σ and a σ-derivation δ on a semisimple ring . Then we study the case of Ore extensions of endomorphism type ($\delta = 0$) showing in particular (Proposition 3.2) that the above mentioned result of Pascaud and Valette can be generalized to the case when σ is an injective endomorphism of R. The general Ore extension $R[x;\sigma,\delta]$ is then analysed, first in the case of a semisimple base ring (Cf. Corollary 3.5). Then it is shown, in Theorem 3.7, that the necessary and sufficient conditions for $R[x;\sigma,\delta]$ to be a PI ring obtained in Section 2 are also valid under the assumption that R is a semiprime PI ring with the ACC on annihilators.

The last section is devoted to the study of the PI property of the Ore extension $R[x;\sigma]$ where σ is an arbitrary endomorphism of R and the coefficient ring R is noetherian. In particular, we give in Theorem 4.7, a necessary and sufficient conditionh for $R[x;\sigma]$ to satisfy a polynomial identity.

1 Preliminaries

Throughout the paper R stands for an associative ring with unity and $Z(R)$ for its center. For any multiplicatively closed subset S of $Z(R)$, RS^{-1} denotes the localization of R with respect to the set of all regular elements from S. In particular, $RZ(R)^{-1}$ is the localization of R with respect to the Ore set of all central regular elements of R.

For a right R-module M, $\mathrm{udim}_R(M)$ denotes its uniform dimension.

In the following proposition we gather classical results which are consequences of a generalized Posner's Theorem and the theorem of Kaplansky (Cf. Rowen's book [15]).

Proposition 1.1. *For a semiprime* PI *ring R, the following conditions are equivalent:*

1. *R is a left (right) Goldie ring;*
2. *R satisfies the* ACC *condition on left (right) annihilators;*
3. *R has finitely many minimal prime ideals;*
4. *R possesses a semisimple classical left (right) quotient ring $Q(R)$ which is equal to the central localization $RZ(R)^{-1}$.*

If R satisfies one of the above equivalent conditions and $\bigoplus_{i=1}^n B_i$ is a decomposition of the semisimple ring $Q(R)$ into simple components, then $\dim_{Z(B_i)} B_i$ is finite, for every $1 \leq i \leq n$. In particular, $\mathrm{udim}_{Z(Q(R))}(Q(R)) < \infty$.

We will use frequently the above proposition without referring to it.

The following observation is probably well-known but we could not find it in the literature:

Proposition 1.2. *Let $B = \bigcup_{i=0}^{\infty} A_i$ be a filtered ring and $\mathrm{gr}(B) = \bigoplus_{i=0}^{\infty} A_i/A_{i-1}$ denote its associated graded ring. If B satisfies a polynomial identity then $\mathrm{gr}(B)$ also satisfies a polynomial identity.*

Proof. Let $G(B)$ denote the Rees ring of B, that is $G(B)$ is a subring of the polynomial ring $B[x]$ consisting of all polynomials $\sum_i a_i x^i \in B[x]$ such that $a_i \in A_i$, for any $i \geq 0$. Then $G(B)$ is a PI ring as a subring of the PI ring $B[x]$. It is known that the ring $G(B)/xG(B)$ is isomorphic to $\mathrm{gr}(B)$ and the thesis follows. □

An Ore extension of a ring R is denoted by $R[x;\sigma,\delta]$, where σ is an endomorphism of R and δ is a σ-derivation, i.e. $\delta\colon R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in R$. Recall that elements of $R[x; \sigma, \delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in R and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$.

The Ore extension $R[x; \sigma, \delta]$ has a natural filtration given by the degree and the associated graded ring is isomorphic to $R[x; \sigma]$. Therefore, by the above proposition, we have:

Corollary 1.3. *If $R[x; \sigma, \delta]$ is a* PI *ring, then $R[x; \sigma]$ also satisfies a polynomial identity.*

Lemma 1.4. *Let σ be an endomorphism of R. Then:*

1. *Suppose that σ is injective and there exists $n \geq 1$ such that $\sigma^n|_{Z(R)}$ is an automorphism of $Z(R)$. Then $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$.*
2. *Suppose that R is a semiprime* PI *ring and σ is injective when restricted to the center $Z(R)$, then σ is injective on R.*
3. *Suppose that R is simple finite dimensional over $Z(R)$ and $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ then σ is an automorphism of R.*

Proof. (1). Let $[a, b]$ denote the commutator of elements $a, b \in R$, i.e. $[a, b] = ab - ba$. Pick $n \geq 1$ such that $\sigma^n|_{Z(R)}$ is an automorphism of $Z(R)$.

Then, for any $r \in R$ and $z \in Z(R)$, we have

$$\sigma^{n-1}([\sigma(z), r]) = [\sigma^n(z), \sigma^{n-1}(r)] = 0.$$

Hence $[\sigma(z), r] \in \ker \sigma^{n-1} = 0$. This gives (1).

The statement (2) is clear, as any nonzero ideal of a semiprime PI ring intersects the center nontrivially.

(3). Since R is simple, $Z(R)$ is a field. Let R' be the left $Z(R)$-linear space R with the action of $Z(R)$ twisted by σ, i.e. $z \cdot r = \sigma(z)r$, for $z \in Z(R)$, $r \in R'$. Now the thesis is a consequence of the fact that $\sigma \colon {}_{Z(R)}R \to_{Z(R)} R'$ is an injective homomorphism of $Z(R)$-linear spaces of the same finite dimension. □

Braun and Hajarnavis showed in [2] that if σ is an injective endomorphism of a prime noetherian PI ring R such that $\sigma|_{Z(R)} = \mathrm{id}_{Z(R)}$, then σ is an automorphism of R. We will see in Example 2.2, that if R is a prime PI ring (so it has the ACC on annihilators), then σ does not have to be onto if $\sigma|_{Z(R)} = \mathrm{id}_{Z(R)}$.

In the lemma below we quote some known results. The first statement is a special case of a result of Jategaonkar (Cf. Proposition 2.4 [7]). The second one is exactly Theorem 3.8 and Proposition 3.6 from [11], respectively.

Lemma 1.5. *Let R be a semiprime left Goldie ring. Suppose that the endomorphism σ is injective. Then:*

1. *Let $\mathcal{C}$ be the set of all regular elements of R. Then $\sigma(\mathcal{C}) \subseteq \mathcal{C}$;*
2. *$R[x;\sigma,\delta]$ is a semiprime left Goldie ring. When R is prime, then $R[x;\sigma,\delta]$ is also a prime ring.*

As we will see in the following result, the above lemma will enable us to reduce some of our considerations to the case when the coefficient ring R is semisimple.

Proposition 1.6. *Let R be a semiprime left Goldie ring. Suppose that σ is an injective endomorphism of R. Then σ and δ can be uniquely extended to the classical ring of quotient $Q(R)$ of R, and the following conditions are equivalent:*

1. *$R[x;\sigma,\delta]$ is a* PI *ring.*
2. *$Q(R)[x;\sigma,\delta]$ is a* PI *ring.*

Proof. By Lemma 1.5(1), $\sigma(\mathcal{C}) \subseteq \mathcal{C}$, where $\mathcal{C}$ is the set of all regular elements of R. This means that σ and δ can be uniquely extended to an injective endomorphism σ and a σ-derivation δ of $Q(R) = \mathcal{C}^{-1}R$ and we can consider the Ore extension $Q(R)[x;\sigma,\delta]$.

The implication (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2). Suppose $R[x;\sigma,\delta]$ satisfies a polynomial identity. Then, by Lemma 1.5(2), $R[x;\sigma,\delta]$ is a semiprime PI left Goldie ring. Thus, $Q(R[x;\sigma,\delta])$ is a semisimple PI ring.

Clearly all elements from the set $\mathcal{C}$ are invertible in $Q(R)[x;\sigma,\delta]$ and every element from $Q(R)[x;\sigma,\delta]$ can be presented in the form $c^{-1}p$ for some $c \in \mathcal{C}$ and $p \in R[x;\sigma,\delta]$. This means that $\mathcal{C}$ is a left Ore set in $R[x;\sigma,\delta]$ and $\mathcal{S}^{-1}(R[x;\sigma,\delta]) = Q(R)[x;\sigma,\delta]$. This yields, in particular, that there exists a natural embedding of $Q(R)[x;\sigma,\delta]$ into the PI ring $Q(R[x;\sigma,\delta])$. This shows that $Q(R)[x;\sigma,\delta]$ satisfies a polynomial identity. $\square$

In the sequel we will need the following:

Lemma 1.7. *Let R be a ring and $\mathcal{S}$ be a right Ore set of regular elements. If M is a right R-module which is $\mathcal{S}$-torsion free then*

$$\operatorname{udim}_R(M) = \operatorname{udim}_{R\mathcal{S}^{-1}}(M \otimes_R R\mathcal{S}^{-1})\,.$$

In particular, if R is a commutative integral domain and M is a torsion free right R-module then $\operatorname{udim}_R(M) = \dim_K(M \otimes_R K)$, where K is the field of fractions of R.

Proof. The particular case comes from [10] Theorem 6.14 and the proof given there can be easily extended to get the first statement. □

In the next lemma we will consider Ore extensions of the form $R[x;\phi]$, where ϕ denotes either an automorphism or a derivation of R. R^ϕ will denote a subring of constants, i.e. $R^\phi = \{x \in R \mid \phi(x) = x\}$ when ϕ is an automorphism and $R^\phi = \{x \in R \mid \phi(x) = 0\}$, when ϕ is a derivation.

Lemma 1.8. *Let R be a ring and $Z = Z(R[x;\phi])$, where ϕ is either an automorphism or a derivation of R. Then:*

1. $\mathrm{udim}_{R^\phi}(R) \leq \mathrm{udim}_Z(R[x;\phi])$.
2. *Suppose that R is a semiprime ring with the* ACC *on annihilators. If $R[x;\phi]$ is a* PI *ring, then:*
   (a) $\mathrm{udim}_{R^\phi}(R) < \infty$.
   (b) *If every regular element of $Z(R)^\phi$ is regular in $Z(R)$, then:*
   $Z(R)Z(R)^{-1} = Z(R)(Z(R)^\phi)^{-1}$ *and* $Q(R) = R(Z(R)^\phi)^{-1}$.

Proof. (1). For $f = \sum_{i=0}^n a_i x^i \in Z$, let $\phi(f)$ denote $\sum_{i=0}^n \phi(a_i)x^i$. Then, $0 = [x, f] = (\phi(f) - f)x$, if ϕ is an automorphism of R. If ϕ is a derivation of R, then $0 = [x, f] = \phi(f)$. The above yields that $Z \subseteq R^\phi[x]$.

For any element $r \in R^\phi$ we have $xr = rx$. Therefore, if M is a right R^ϕ-submodule of R, then $M[x]$ is a right $R^\phi[x]$-submodule of $R[x;\phi]$. In particular, $M[x]$ has also a structure of Z-module, as $Z \subseteq R^\phi[x]$. One can easily check that direct sums of R^ϕ-submodules of R lift to direct sums of $R^\phi[x]$-submodules of $R[x;\phi]$. Therefore $\mathrm{udim}_{R^\phi}(R) \leq \mathrm{udim}_{R^\phi[x]}(R[x;\phi]) \leq \mathrm{udim}_Z(R[x;\phi])$.

(2)(a). Suppose that $R[x,\phi]$ is a PI ring and the coefficient ring R is semiprime with the ACC condition on annihilators. Then R also satisfies a polynomial identity, so R is a semiprime PI left Goldie ring. Therefore, by Lemma 1.5, $R[x,\phi]$ is a semiprime Goldie PI ring with a semisimple quotient ring $Q(R[x,\phi]) = R[x,\phi]Z^{-1}$.

Making use of the statement (1), Lemma 1.7 and Proposition 1.1, we obtain:

$$\mathrm{udim}_{R^\phi}(R) \leq \mathrm{udim}_Z(R[x;\phi]) = \mathrm{udim}_{ZZ^{-1}}(Q(R[x;\phi])) < \infty.$$

This gives the statement (a).

(b). Suppose that every regular element of $Z(R)^\phi$ is regular in $Z(R)$. That is $B = Z(R)(Z(R)^\phi)^{-1}$ and $Z(R)^\phi(Z(R)^\phi)^{-1}$ means localizations with respect the same Ore set of all regular elements of $Z(R)^\phi$.

Notice that, in order to prove the statement (b), it is enough to show that $Z(R)Z(R)^{-1} = B$, as $Q(R) = RZ(R)^{-1}$. Since the ACC on annihilators is a hereditary condition on subrings and $Z(R)$ is a reduced ring (i.e. $Z(R)$ does not contain nontrivial nilpotent elements), $Z(R)^\phi$ is a commutative reduced ring with the ACC on annihilators. Therefore, its classical quotient ring $A = Z(R)^\phi(Z(R)^\phi)^{-1} \subseteq B$ is a finite product of fields, say $A = \bigoplus_{i=1}^n K_i$, where $K_i = e_iA$ for suitable primitive orthogonal idempotents, $1 \leq i \leq n$.

Recall that ϕ is either an automorphism or a derivation of R. Hence $Z(R)$ is stable by ϕ and we can consider the Ore extension $Z(R)[x;\phi]$ and, since $R[x;\phi]$ is a PI ring, $Z(R)[x;\phi]$ is also a PI ring . Now, we can apply the statement $(2)(a)$ to $Z(R)$ and get $\mathrm{udim}_{Z(R)^\phi}(Z(R)) < \infty$. Consequently, by Lemma 1.7, we get $\mathrm{udim}_A(B) < \infty$. This implies that, for any $1 \leq i \leq n$, e_iB is a finite dimensional algebra over the field $K_i = e_iA$. Therefore every regular element in e_iB is invertible in e_iB. This, in turn, implies that every regular element of $B = \bigoplus_{i=1}^n e_iB$ is invertible in $B = Z(R)(Z(R)^\phi)^{-1}$. This shows that B is equal to $Z(R)Z(R)^{-1}$ and completes the proof of the lemma. □

Remark 1.9. Let us observe that if R is a prime ring, then the assumption of the statement (2b) from the above proposition is always satisfied.

2 Prime Coefficient Ring

In this section R will stand for a prime PI ring. We first continue to gather some information on the behaviour of σ on the center.

Proposition 2.1. *Let R be a prime* PI *ring and σ an endomorphism of R such that σ^n is an automorphism of the center $Z(R)$ of R. Then:*

1. *σ extends uniquely to an automorphism of the localization $RZ(R)^{-1}$.*
2. *Suppose additionally that $\sigma^n|_{Z(R)} = \mathrm{id}_{Z(R)}$. Then there is $0 \neq u \in Z(R)$ such that $\sigma(u) = u$, σ is an automorphism of RS^{-1} and σ^n is an inner automorphism of RS^{-1}, where $S = \{u^k \mid k \geq 0\}$.*

Proof. (1) By Lemma 1.4, σ is injective and $\sigma(Z(R)) = Z(R)$. Thus σ has a unique extension to the localization $RZ(R)^{-1}$ which, by Posner's Theorem, is a simple, finite dimensional algebra over the center $Z(R)Z(R)^{-1}$. Now the statement (1) is a direct consequence of Lemma 1.4(3).

(2) By (1), σ is an automorphism of $RZ(R)^{-1}$ and the theorem of Skolem-Noether implies that σ^n is an inner automorphism of $RZ(R)^{-1}$. Therefore, one can choose a regular element $a \in R$ with the inverse $bv^{-1} \in RZ(R)^{-1}$ such that $\sigma^n(r) = arbv^{-1}$ for all $r \in R$. Let $u = v\sigma(v)\ldots\sigma^{n-1}(v)$. Then $\sigma(u) = u$, the element bu^{-1} has the inverse $a\sigma(v)\ldots\sigma^{n-1}(v)$ and also determines σ^n. This yields the thesis. □

In the sequel we will say that an automorphism σ of a ring R is of finite inner order if σ^n is an inner automorphism of R, for some $n \geq 1$.

At this point we make a small digression not directly related to the main theme of the paper. It is known (Cf. [9]) that for any ring R with a fixed injective endomorphism σ there exists a universal over-ring A of R, called a Jordan extension of R, such that σ extends to an automorphism of A and $A = \bigcup_{i=0}^{\infty} \sigma^{-i}(R)$. In this case we will write $R \subseteq_\sigma A$.

It is easy to check that if σ becomes an inner automorphism of A, then $R = A$. Also, if R is a prime PI ring, then A is prime PI as well.

Recall that an automorphism of a prime PI ring R is X-inner if and only if it becomes inner when extended to the classical quotient ring $Q(R) = RZ(R)^{-1}$.

Suppose that R is a prime PI ring and σ is an endomorphism of R such that $\sigma^n|_{Z(R)} = \mathrm{id}|_{Z(R)}$. Then, by Proposition 2.1, $R \subseteq_\sigma A \subseteq RS^{-1}$, where S consists of powers of a single central element. Moreover σ is an X-inner automorphism of A. The following example shows, that all inclusions $R \subseteq_\sigma A \subseteq RS^{-1}$ can be strict. This example will be used again later in the paper.

Example 2.2. Let $R = \begin{bmatrix} \mathbb{Z} + x\mathbb{Q}[x] & \mathbb{Z} + x\mathbb{Q}[x] \\ x\mathbb{Q}[x] & \mathbb{Z} + x\mathbb{Q}[x] \end{bmatrix}$ be a subring of $M_2(\mathbb{Q}[x])$ and σ denote the inner automorphism of $M_2(\mathbb{Q}[x])$ adjoint to the element $u = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $\sigma(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & 2b \\ \frac{1}{2}c & d \end{bmatrix}$. This means that the restriction of σ to R is an injective endomorphism of R which is not onto. One can check that $R \subseteq_\sigma A = \begin{bmatrix} \mathbb{Z} + x\mathbb{Q}[x] & \mathbb{Z}[\frac{1}{2}] + x\mathbb{Q}[x] \\ x\mathbb{Q}[x] & \mathbb{Z} + x\mathbb{Q}[x] \end{bmatrix}$ and σ becomes an inner automorphism on the localization RS^{-1}, where S denotes the multiplicatively closed set generated by 2.

Definition 2.3. Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. We say that the center of the Ore extension $R[x; \sigma, \delta]$ is nontrivial if it contains a nonconstant polynomial.

Lemma 2.4. *Let σ be an injective endomorphism of a prime ring R. Then:*

1. *If $R[x; \sigma, \delta]$ is a* PI *ring, then the center $Z(R[x; \sigma, \delta])$ is nontrivial.*
2. *Let $f = ax^n + ... \in Z(R[x; \sigma, \delta])$ be a polynomial of degree $n \geq 1$. Then a is a regular element of R, $\sigma^n|_{Z(R)} = \mathrm{id}_{Z(R)}$ and $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$.*

Proof. (1). Suppose that $R[x; \sigma, \delta]$ is a PI ring. Then Lemma 1.5 and Proposition 1.1 imply that $R[x; \sigma, \delta]$ is a prime PI ring. Thus every essential one-sided ideal contains a nonzero central element. Since σ is injective, the element x is regular in $R[x;\sigma,\delta]$. Therefore $R[x;\sigma,\delta]x$ contains a nonzero central element $f = ax^n + a_{n-1}x^{n-1} + \ldots + a_1x$, where $a \neq 0$ and $n \geq 1$ and (1) follows.

(2) Let $f = ax^n + \ldots \in Z(R[x;\sigma,\delta])$ be such that $a \neq 0$ and $n \geq 1$. Making use of $xf = fx$ and $rf = fr$, for any $r \in R$, we obtain $\sigma(a) = a$ and $ra = a\sigma^n(r)$, for any $r \in R$.

We claim that a is a regular element in R. Indeed, if $b \in R$ is such that $ba = 0$, then $bRa = ba\sigma^n(R) = 0$. Hence $b = 0$, as R is a prime ring. Thus a is left regular. If $ab = 0$, then $0 = \sigma^n(a)\sigma^n(b) = a\sigma^n(b) = ba$. Since a is left regular, $b = 0$ follows.

Now, for any $z \in Z(R)$ we have $az = za = a\sigma^n(z)$. Thus $a(z - \sigma^n(z)) = 0$ and $\sigma^n(z) = z$, for any $z \in Z(R)$, follows. The last assertion of (2) is then a consequence of Lemma 1.4(1). □

Proposition 2.5. *Let R be a prime* PI *ring and σ an injective endomorphism of R. The following conditions are equivalent:*

1. *$R[x;\sigma]$ is a* PI *ring.*
2. *$\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ of finite order.*
3. *There exists $0 \neq u \in Z(R)$ such that $\sigma(u) = u$ and σ is an automorphism of finite inner order of the localization RS^{-1} , where S denotes the set of all powers of u.*

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are given by Lemma 2.4 and Proposition 2.1, respectively.

(3) ⇒ (1). Suppose that (3) holds and let σ^n, $n \geq 1$, be an inner automorphism of RS^{-1}. By the choice of S, the set S is central in $R[x;\sigma]$ and $(R[x;\sigma])S^{-1} = RS^{-1}[x;\sigma]$. Since σ^n is an inner automorphism of RS^{-1}, the subring $RS^{-1}[x^n] \subseteq RS^{-1}[x;\sigma]$ is isomorphic to the usual polynomial ring $RS^{-1}[y]$, so it satisfies a polynomial identity, as RS^{-1} is a PI ring. Now, the fact that $RS^{-1}[x;\sigma]$ is a finitely generated free module over the PI subring $RS^{-1}[x^n]$ implies that $R[x;\sigma]$ satisfies a polynomial identity. □

In case σ is an automorphism of R, the equivalence (1) ⇔ (2) in the above proposition was also obtained by Pascaud and Valette (Cf. [14]) using another approach.

Before stating the next results we need to recall some definitions (Cf.[12]):

Definition 2.6. Let R be a ring, σ an endomorphism and δ a σ-derivation of R, respectively. We say that:

1. δ is quasi algebraic if there exists $n \geq 1$ and elements $b, a_1, \dots, a_n \in R$, with $a_n \neq 0$, such that $\sum_{i=1}^{n} a_i \delta^i = \delta_{b,\sigma^n}$ where δ_{b,σ^n} denotes the inner σ^n-derivation adjoint to the element b, that is $\delta_{b,\sigma^n}(r) = br - \sigma^n(r)b$, for any $r \in R$.

2. A polynomial $p \in R[x; \sigma, \delta]$ is right semi-invariant if for any element $a \in R$ there exists $b \in R$ such that $pa = bp$.

Theorem 2.7. *Suppose σ is an injective endomorphism of a prime* PI *ring R. Let $Q(R) = RZ(R)^{-1}$ denote the classical quotient ring of R. The following conditions are equivalent:*

1. *$R[x; \sigma, \delta]$ is a* PI *ring.*
2. *There exists a nonconstant central polynomial in $R[x; \sigma, \delta]$ with a regular leading coefficient.*
3. *The center of $R[x; \sigma, \delta]$ is nontrivial.*
4. *The center of $Q(R)[x; \sigma, \delta]$ is nontrivial.*
5. *There exists a nonconstant central polynomial in $Q(R)[x; \sigma, \delta]$ with invertible leading coefficient.*
6. *$Q(R)[x; \sigma, \delta]$ is a* PI *ring.*
7. *σ is an automorphism of $Q(R)$ of finite inner order and δ is a quasi algebraic σ-derivation of $Q(R)$.*
8. *σ is an automorphism of $Q(R)$ of finite inner order and $Q(R)[x; \sigma, \delta]$ contains a monic nonconstant semi-invariant polynomial.*

Proof. The implication (1) $\Rightarrow$ (2) is given by Lemma 2.4. The implication (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Suppose that the center of $R[x; \sigma, \delta]$ is nontrivial. Then, by Lemma 2.4(2), $\sigma(Z(R)) = Z(R)$. This means that we can extend uniquely σ and δ to an endomorphism and a σ-derivation of $Q(R) = RZ(R)^{-1}$, respectively, and consider the over-ring $Q(R)[x; \sigma, \delta]$ of $R[x; \sigma, \delta]$. It is standard to check that $Z(R[x; \sigma, \delta]) \subseteq Z(Q(R)[x; \sigma, \delta])$. This shows that the center of $Q(R)[x; \sigma, \delta]$ is nontrivial.

The implication (4) $\Rightarrow$ (5) is given by Lemma 2.4(2) applied to the ring $Q(R)$ and the fact that every regular element of $Q(R)$ is invertible.

(5) $\Rightarrow$ (6). Let $f \in Q(R)[x; \sigma, \delta]$ be a central polynomial of degree $n \geq 1$ with an invertible leading coefficient. Then the subring $Q(R)[f] \subseteq Q(R)[x; \sigma, \delta]$

is isomorphic to the usual polynomial ring $Q(R)[y]$ in one indeterminate y, so the ring $Q(R)[f]$ satisfies a polynomial identity. Notice also that, due to Lemma 2.4(2) and Proposition 2.1, σ is an automorphism of $Q(R)$. Now, the fact that $Q(R)[x;\sigma,\delta]$ is a free module over $Q(R)[f]$ with basis $1, x, \ldots, x^{n-1}$ implies that $Q(R)[x;\sigma,\delta]$ is a PI ring.

The implication (6) $\Rightarrow$ (1) is obvious.

The above shows that conditions (1) $\div$ (6) are equivalent and that the extension of σ to $Q(R)$ is an automorphism of $Q(R)$.

Now the equivalence of statements (4), (7) and(8) is given by Theorem 3.6 in [12]. $\square$

Remark 2.8. (1). Notice that due to Proposition 2.1(2), the assumption in (7) and (8) of the above theorem that σ^n is an inner automorphism of $Q(R)$, for some $n \geq 1$, can be replaced by a condition that $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ of finite order.

(2). One can also replace the ring $Q(R)$ in the above theorem by a localization RS^{-1} where S denotes a multiplicatively closed set consisting of all powers of a suitably chosen central σ-invariant element.

Theorem 2.7 says that if $R[x;\sigma,\delta]$ is a PI ring, then the extension of σ to $Q(R)$ has to be an automorphism of $Q(R)$. It is not a surprise as, by Corollary D[2], every endomorphism of a semiprime Noetherian PI ring R which is identity on $Z(R)$ is always an automorphism of R. Nevertheless, when R is a prime PI ring (so it satisfies the ACC on annihilators), then the injective endomorphism σ of R does not have to be onto when $R[x;\sigma,\delta]$ is a PI ring.

Example 2.9. Let R and σ be as in Example 2.2. Then R is a prime ring, the injective endomorphism σ is not onto and $R[y;\sigma]$ satisfies a polynomial identity, since $R[y;\sigma] \subseteq M_2(Q[x])[y;\sigma] \simeq M_2(Q[x])[z]$.

Theorem 2.10. *Let R be a prime ring, σ an injective endomorphism and δ a σ-derivation of R. Then $R[x;\sigma,\delta]$ is a* PI *ring if and only if R is* PI*, $\sigma|_{Z(R)}$ is an automorphism of $Z(R)$ of finite order and one of the following conditions holds:*

1. *If* $\mathrm{char} R = 0$,
$$Q(R)[x;\sigma,\delta] \simeq \begin{cases} Q(R)[x;\sigma] & if \quad \sigma|_{Z(R)} \neq \mathrm{id}_{Z(R)} \\ Q(R)[x] & else \end{cases}$$
2. *If* $\mathrm{char} R = p \neq 0$,
$$Q(R)[x;\sigma,\delta] \simeq \begin{cases} Q(R)[x;\sigma] & if \quad \sigma|_{Z(R)} \neq \mathrm{id}_{Z(R)} \\ Q(R)[x;d] & else \end{cases}$$

where d is a suitable derivation of $Q(R) = RZ(R)^{-1}$ such that:

(a) $d(R) \subseteq R$

(b) there exist elements $q_l = 1, \dots, q_1 \in Q(R)$ such that $\sum_{i=0}^{l} q_i d^{p^i}$ is an inner derivation of $Q(R)$

Proof. Suppose $R[x;\sigma,\delta]$ is a PI ring. Then R is PI and Theorem 2.7 and Remark 2.8 show that $Q(R)[x;\sigma,\delta]$ is a PI ring, σ is an automorphism of $Q(R)$ and $\sigma|_{Z(R)}$ is an automorphism of the center $Z(R)$ of finite order.

Suppose that $\sigma|_{Z(R)} \neq \mathrm{id}|_{Z(R)}$ and let $c \in Z(R)$ be such that $\sigma(c) \neq c$. Then it is well-known that δ is an inner σ-derivation of $Q(R)$ adjoint to the element $a = (c - \sigma(c))^{-1}\delta(c)$. Then $Q(R)[x;\sigma,\delta] = Q(R)[x - a;\sigma] \simeq Q(R)[x;\sigma]$.

Suppose $\sigma|_{Z(R)} = \mathrm{id}_{Z(R)}$. Then, by Proposition 2.1(2), σ is an inner automorphism of $Q(R)$, say induced by an invertible element $c \in Q(R)$, i.e. $\sigma(r) = crc^{-1}$, for any $r \in Q(R)$. Since $Q(R)$ is a central localization of R we can write $c^{-1} = uz^{-1}$ for some $u \in R$ and $z \in Z(R)$ and we have $\sigma(r) = u^{-1}ru$ for all $r \in Q$. Then $u\delta = d$ is a derivation of $Q(R)$ such $d(R) \subseteq R$ and $Q(R)[x;\sigma,\delta] = Q(R)[ux;\mathrm{id},u\delta] \simeq Q(R)[x;d]$.

Applying Theorem 2.7 to the PI ring $Q(R)[x;d]$ we know that this ring contains a monic nonconstant semi-invariant polynomial. Now the thesis is a consequence of Proposition 2.8 and Lemma 2.3 from [12] and the fact that $Q(R)[x;d]$ is isomorphic to $Q(R)[x]$, provided d is an inner derivation.

Conversely, if R is a prime PI ring and σ^n is the identity on the center, then Theorem 2.7(8), Remark 2.8 and the hypothesis made on $Q(R)[x;\sigma,\delta]$ shows that $R[x,\sigma,\delta]$ is PI. □

Remark 2.11. In case σ is an automorphism of R, the statement (1) from the above Theorem is exactly the result of Jondrup [8], see also the book by Goodearl and Brown [3].

We have seen that the center $Z(R)$ of R plays a crucial role in determining if an Ore extension satisfies a polynomial identity. This theme will be pursued further in the next result and in Section 3.

For any subring A of R, $A^{\sigma,\delta}$ will denote the the subalgebra of (σ,δ)-constants, i.e. $A^{\sigma,\delta} = \{a \in A \mid \sigma(a) = a \text{ and } \delta(a) = 0\}$. Notice that we do not require that A is σ or δ stable.

With the above notations we have :

Theorem 2.12. *Suppose that σ is an injective endomorphism of a prime* PI *ring R with the center Z. Let K denote the field of fractions of $Z^{\sigma,\delta}$. The following conditions are equivalent:*

1. *$R[x;\sigma,\delta]$ is a* PI *ring.*

2. $\operatorname{udim}_{Z^{\sigma,\delta}}(Z) < \infty$

3. $\dim_K Q(R) < \infty$ *and* $Q(R) = R(Z^{\sigma,\delta})^{-1}$.

4. $\dim_K Q(R) < \infty$.

Proof. (1) $\Rightarrow$ (2) Suppose that $R[x;\sigma,\delta]$ is a PI ring. Then, by Theorem 2.10, $Q(R)[x;\sigma,\delta]$ is $Q(R)$-isomorphic to $Q(R)[x;\phi]$, where either $\phi = \sigma$ and $\phi(Z) = Z$ or ϕ is a derivation of $Q(R)$ such that $\phi(R) \subseteq R$. In particular, in any case we have $\phi(Z) \subseteq Z$ and we can consider $Z[x;\phi]$ as a subring of the PI ring $Q(R)[x,\phi]$. Now, Lemma 1.8(2) shows that $\operatorname{udim}_{Z^\phi}(Z)$ is finite.

Notice that:

$$Z(Q(R))^{\sigma,\delta} = Q(R) \cap Z(Q(R)[x;\sigma,\delta]) = Q(R) \cap Z(Q(R)[x;\phi]) = Z(Q(R))^{\phi}.$$

Therefore, $Z^{\sigma,\delta} = Z \cap Z(Q(R))^{\sigma,\delta} = Z \cap Q(R)^{\phi} = Z^{\phi}$ and $\operatorname{udim}_{Z^{\sigma,\delta}}(Z) = \operatorname{udim}_{Z^\phi}(Z) < \infty$ follows.

(2) $\Rightarrow$ (3). Suppose that $\operatorname{udim}_{Z^{\sigma,\delta}}(Z) < \infty$. Then, making use of Lemma 1.7, we obtain: $\dim_K Z(Z^{\sigma,\delta})^{-1} = \operatorname{udim}_{Z^{\sigma,\delta}}(Z) < \infty$. This implies that the commutative domain $Z(Z^{\sigma,\delta})^{-1}$ is a field. Therefore $Z(Z^{\sigma,\delta})^{-1} = ZZ^{-1}$ and $Q(R) = RZ^{-1} = R(Z^{\sigma,\delta})^{-1}$. Since $Q(R)$ is finite dimensional over its center $ZZ^{-1} = Z(Z^{\sigma,\delta})^{-1}$ which is a finite dimensional field extension of K, we have $\dim_K Q(R) < \infty$.

The implication (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). Under the hypothesis (4), $Q(R)[x;\sigma,\delta]$ is a finitely generated module over the commutative polynomial ring $K[x]$. This yields that $Q(R)[x;\sigma,\delta]$ is a PI ring and so is $R[x;\sigma,\delta]$. $\square$

3 Semiprime Coefficient Ring

In this section we will investigate Ore extensions $R[x;\sigma,\delta]$ over a semiprime coefficient ring R satisfying the ACC on annihilators. We will frequently use the following lemma, which is an obvious application of results of Cauchon and Robson from [5] (Cf. Lemma 1.1 to Lemma 1.4).

Lemma 3.1. *Let $R = \bigoplus_{i=1}^{s} B_i$ be a decomposition of a semisimple ring R into its simple components and let σ, δ be an injective endomorphism and a σ-derivation of R, respectively. Then :*

1. *There exists a permutation ρ of the index set $\{1, \ldots, s\}$ such that $\sigma(B_i) \subseteq B_{\rho(i)}$ and $\delta(B_i) \subseteq B_i + B_{\rho(i)}$.*

2. *If* $\{1,\dots,s\} = \cup_{j=1}^{k}\mathcal{O}_j$ *is the decomposition of the index set into orbits under the action of the permutation* ρ *and* $A_j := \bigoplus_{i\in\mathcal{O}_j} B_i$*, then* $R[x;\sigma,\delta] = \bigoplus_{j=1}^{k} A_j[x_j;\sigma|_{A_j},\delta|_{A_j}]$.
3. *Let* $j \in \{1,\dots,k\}$ *be such that* $|\,\mathcal{O}_j\,| > 1$*, then* $\delta|_{A_j}$ *is an inner* $\sigma|_{A_j}$*-derivation of* A_j*. In particular,* $A_j[x_j;\sigma|_{A_j},\delta|_{A_j}]$ *is* A_j*-isomorphic to* $A_j[x_j;\sigma|_{A_j}]$.
4. *There exists an* $m \geq 1$ *such that* $\sigma^m(B_i) \subseteq B_i$*, for all* $i \in \{1,\dots,s\}$.

Let us first consider the case of an Ore extension of endomorphism type.

Proposition 3.2. *Let* σ *be an injective endomorphism of a semiprime* PI *ring* R *satisfying the* ACC *on annihilators. The following conditions are equivalent:*

1. $R[x;\sigma]$ *is a* PI *ring.*
2. *The restriction* $\sigma|_{Z(R)}$ *is an automorphism of* $Z(R)$ *of finite order.*
3. *There exists* $n \geq 1$ *such that* σ^n *is identity on the center* $Z(R)$ *of* R*.*
4. *There exists a regular element* $u \in Z(R)$ *such that* $\sigma(u) = u$ *and* σ *is an automorphism of the localization* RS^{-1} *of finite inner order, where* S *denotes the set of all powers of* u*.*
5. σ *is an automorphism of* $Q(R)$ *of finite inner order.*

Proof. By Lemma 1.5, σ can be extended to an injective endomorphism of $Q(R) = RZ(R)^{-1}$. Let $Q(R) = \bigoplus_{i=1}^{s} B_i$ be a decomposition of $Q(R)$ into its simple components.

(1) $\Rightarrow$ (5). Suppose $R[x;\sigma]$ is a PI ring. Thus, by Proposition 1.6, $Q(R)[x;\sigma]$ also satisfies a polynomial identity. By Lemma 3.1, there exists $m \geq 1$, such that all components B_i are σ^m-stable. Therefore, $Q(R)[x;\sigma^m] \simeq Q(R)[x^m;\sigma^m] \subseteq Q(R)[x;\sigma]$ is also a PI ring and

$$Q(R)[x;\sigma^m] = (\bigoplus_{i=1}^{s} B_i)[x;\sigma^m] \simeq \bigoplus_{i=1}^{s} B_i[x_i;\sigma^m].$$

This shows that, for any $1 \leq i \leq s$, $B_i[x_i;\sigma^m]$ satisfies a polynomial identity and Theorem 2.7(8) applied to each simple component B_i yields that there exists $k \geq 1$ such that σ^n, where $n = mk$, is an inner automorphism of $Q(R)$, i.e. (5) holds.

Since $Q(R) = RZ(R)^{-1}$, the implication (5) $\Rightarrow$ (4) can be proved using the same argument as in the proof of Proposition 2.1(2).

The implication (4) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (2) is a direct consequence of Lemma 1.4(1).

(2) $\Rightarrow$ (1). Suppose that (2) holds and let n denote the order of $\sigma|_{Z(R)}$. Then $\sigma^n(B_i) \subseteq B_i$, for $1 \leq i \leq s$. Then, by Lemma 1.4 and the theorem of Skolem-Noether, $\sigma^n|_{B_i}$ is an inner automorphism of B_i, for any i. Hence σ^n is an inner automorphism of $Q(R)$ and the subring $Q(R)[x^n] \subseteq Q(R)[x;\sigma]$ is isomorphic to a polynomial ring $Q(R)[y]$, so it satisfies a polynomial identity. This implies that $Q(R)[x;\sigma]$ is a PI ring, as $Q(R)[x;\sigma]$ is a finitely generated free module over its subring $Q(R)[x^n]$ and (1) follows. $\square$

As an immediate application of the above proposition, Corollary 1.3 and Proposition 1.6 we easily get the following:

Corollary 3.3. *Let R be a semiprime ring with* ACC *on annihilators and σ an injective endomorphism of R. If $R[x;\sigma,\delta]$ is a* PI *ring then $Q(R)[x;\sigma,\delta]$ is a* PI *ring and σ is an automorphism of the semisimple ring $Q(R)$ of finite inner order.*

Thus while investigating the PI property of $R[x;\sigma,\delta]$, the crucial case is when the ring R is semisimple. Moreover, by Lemma 3.1(2), one may restrict attention to the case when σ acts transitively on the set of all simple components B_i of $R = \bigoplus_{i=1}^{s} B_i$.

Proposition 3.4. *Let $R = \bigoplus_{i=1}^{s} B_i$ be a decomposition of a semisimple* PI *ring into simple components and σ an injective endomorphism of R. Suppose that σ acts transitively on the set of simple components. Then the following conditions are equivalent:*

1. *$R[x;\sigma,\delta]$ is a* PI *ring.*
2. *σ is an automorphism of R of finite inner order and one of the following conditions holds:*
   - *(a) $R[x;\sigma,\delta]$ is isomorphic either to $R[x;\sigma]$ or to $R[x]$.*
   - *(b) R is a simple ring of a nonzero characteristic p and $R[x;\sigma,\delta] \simeq R[x;d]$ where d is a derivation of R such that there exist elements $q_l = 1, \ldots, q_1 \in R$ such that $\sum_{i=0}^{l} q_i d^{p^i}$ is an inner derivation of R.*

Proof. If R is simple, i.e. $R = B_1$, then the proposition is a direct consequence of Theorem 2.10.

If R is not simple then, by Lemma 3.1(3), $R[x;\sigma,\delta] \simeq R[x;\sigma]$ and the proposition is a consequence of Proposition 3.2. $\square$

Corollary 3.5. *Let R be a semisimple* PI *ring with an injective endomorphism σ. The following conditions are equivalent:*

1. $R[x;\sigma,\delta]$ *is a* PI *ring.*
2. *The center of $R[x;\sigma,\delta]$ contains a nonconstant polynomial with invertible leading coefficient.*
3. $\mathrm{udim}_{Z(R)^{\sigma,\delta}}(Z(R))$ *is finite.*

Proof. (2) $\Rightarrow$ (1). Let $f \in R[x;\sigma,\delta]$ be a nonconstant polynomial from the center of $R[x;\sigma,\delta]$ with invertible leading coefficient. Then the subring $R[f] \subseteq R[x;\sigma,\delta]$ satisfies a polynomial identity and $R[x;\sigma,\delta]$ is a finitely generated left module over $R[f]$. This implies that $R[x;\sigma,\delta]$ is a PI ring.

(1) $\Rightarrow$ (2). Let $R = \bigoplus_{i=1}^{s} B_i = \bigoplus_{j=1}^{k} A_j$, where $A_j = \bigoplus_{i\in\mathcal{O}_j} B_i$, be a decomposition of R described in Lemma 3.1. Then, by the same lemma, we have $R[x;\sigma,\delta] = \bigoplus_{j=1}^{k} A_j[x_j;\sigma_j,\delta_j]$, where $\sigma_j = \sigma|_{A_j}$ and $\delta_j = \delta|_{A_j}$. Hence, by Proposition 3.4, the PI ring $T_j = A_j[x_j;\sigma_j,\delta_j]$ is A_j-isomorphic to one of the following rings: $A_j[x]$, $A_j[x;\sigma_j]$, where σ_j is an automorphism of A_j of finite inner order or $A_j[x;d_j]$, where d_j is a derivation of a simple ring A_j.

Suppose $T_j \simeq A_j[x;\sigma_j]$ and let $n \geq 1$ and $u \in A_j$ be an invertible element such that, for any $a \in T_j$, $\sigma^n(a) = u^{-1}au$. It is known that, eventually replacing n by n^2 and u by $u\sigma(u)\ldots\sigma^{n-1}(u)$, we may additionally assume that $\sigma(u) = u$. Then $f_j = ux^n$ is a polynomial from the center of T_j.

Suppose that $T_j \simeq A_j[x;d_j]$, where A_j is a simple ring. Then, by Theorem 2.7, T_j contains a nonconstant central polynomial f_j with an invertible leading coefficient.

The above shows that, for any $1 \leq j \leq k$, $T_j = A_j[x_j;\sigma_j,\delta_j]$ contains a nonconstant central polynomial f_j with invertible leading coefficient. Since a power of a central element is again central, we may choose the polynomials f_j's in such a way that $\deg f_i = \deg f_j$, for all $1 \leq i,j \leq k$. Then the polynomial $f = \sum_{j=1}^{k} f_j$ belongs to the center of $R[x;\sigma,\delta]$, is nonconstant and the leading coefficient of f is invertible.

(1) $\Leftrightarrow$ (3). We will continue to use the notation as in the proof of (1) $\Rightarrow$ (2). Notice that $Z(R) = \bigoplus_{j=1}^{k} Z(A_j)$ and $Z(R)^{\sigma,\delta} = \bigoplus_{j=1}^{k} Z(A_j)^{\sigma_j,\delta_j}$. Hence $\mathrm{udim}_{Z(R)^{\sigma,\delta}}(Z(R)) = \sum_{j=1}^{k} \mathrm{udim}_{Z(A_j)^{\sigma_j,\delta_j}}(Z(A_j))$. This means that, without loosing generality, we may assume that $R = A_1$, i.e. σ acts transitively on the simple components of R.

If R is simple, then the equivalence (1) $\Leftrightarrow$ (3) is given by Theorem 2.12.

Suppose R is not simple. Then, by Lemma 3.1(3), $R[x;\sigma,\delta]$ is R-isomorphic to $R[x;\sigma]$. Since $Z(R)^{\sigma,\delta} = Z(R[x;\sigma,\delta]) \cap R$ and $Z(R)^{\sigma} = Z(R[x;\sigma]) \cap R$, we may replace $R[x;\sigma,\delta]$ by $R[x;\sigma]$.

If $R[x;\sigma]$ satisfies a polynomial identity then Lemma 1.8(2a), applied to $Z(R)[x;\sigma]$, shows that $\operatorname{udim}_{Z(R)^\sigma}(Z)<\infty$.

Suppose now, that $\operatorname{udim}_{Z(R)^\sigma}(Z)$ is finite. Recall that $R=\bigoplus_{i=1}^s B_i$ and, by Lemma 3.1(1), there is $n\geq 1$, such that $\sigma^n(B_i)\subseteq B_i$, for all i and $R[x;\sigma^n]=\bigoplus_{i=1}^s B_i[x_i;\tau_i]$, where $\tau_i=\sigma^n|_{B_i}$. Since $Z(R)^\sigma\subseteq Z(R)^{\sigma^n}$, $\operatorname{udim}_{Z(R)^{\sigma^n}}(Z)<\infty$ and, consequently, $\operatorname{udim}_{Z(B_i)^{\tau_i}}(Z(B_i))<\infty$, for any $1\leq i\leq s$. Therefore, Theorem 2.12 applied to Ore extensions $B_i[x_i;\tau_i]$ shows that $R[x;\sigma^n]=\bigoplus_{i=1}^s B_i[x_i;\tau_i]$ is a PI ring. Since $R[x;\sigma]$ is a finitely generated module over its subring $R[x^n]$ which is itself isomorphic to the PI ring $R[x;\sigma^n]$, we conclude that $R[x;\sigma]$ is a PI ring. □

The following lemma is of crucial importance for the forthcoming theorem.

Lemma 3.6. *Suppose that the ring $R[x;\sigma,\delta]$ satisfies a polynomial identity, where R is a semiprime ring with the* ACC *on annihilators and σ is an injective endomorphism of R. Let Z denote the center of R and $Q=Q(R)$. Then:*

1. *$Q=R(Z^{\sigma,\delta})^{-1}$*
2. *If an element $a\in Z^{\sigma,\delta}$ is regular in $Z^{\sigma,\delta}$, then a is regular in R.*
3. *$Z(Q)=Z(Z^{\sigma,\delta})^{-1}$ and $Z(Q)^{\sigma,\delta}=Z^{\sigma,\delta}(Z^{\sigma,\delta})^{-1}$*

Proof. By Proposition 1.6, we can extend σ and δ to the classical semisimple quotient ring $Q=Q(R)$ of R. Let $Q=\bigoplus_{i=1}^s B_i=\bigoplus_{j=1}^k A_j$, where $A_j=\bigoplus_{i\in\mathcal{O}_j} B_i$, be a decompositions of Q described in Lemma 3.1. Then, by the same lemma, we have $Q[x;\sigma,\delta]=\bigoplus_{j=1}^k A_j[x_j;\sigma_j,\delta_j]$, where $\sigma_j=\sigma|_{A_j}$ and $\delta_j=\delta|_{A_j}$.

(1). Since $Q=RZ^{-1}$, in order to show that $Q=R(Z^{\sigma,\delta})^{-1}$, it is enough to prove that any regular element z from the center of R is invertible in $R(Z^{\sigma,\delta})^{-1}$.

Let $z\in Z$ be regular in Z. By Corollary 3.3, $\sigma|_Z$ is an automorphism of Z of finite order $n\geq 1$. Then, the element $w=z\sigma(z)\cdots\sigma^{n-1}(z)\in Z$ is regular and $\sigma(w)=w$. From this we easily deduce that z is invertible in $R(Z^\sigma)^{-1}$. This means that $Q=R(Z^\sigma)^{-1}$. Therefore, we may assume that our regular element z belongs to Z^σ.

Recall that $Q=\bigoplus_{j=1}^k A_j$, where $\sigma(A_j)\subseteq A_j$ and $\delta(A_j)\subseteq A_j$, for all j. Thus, in particular, $Z(Q)^\sigma=\bigoplus_{j=1}^k Z(A_j)^{\sigma_j}$ and we can present our element z in the form $z=z_1+\cdots+z_k$, where $z_j\in Z(A_j)^{\sigma_j}$, for $1\leq j\leq k$.

Notice that Lemma 3.1(3) (when A_j is not simple) and Theorem 2.10 (when A_j is a simple ring) imply that the restriction $\delta|_{A_j}=\delta_j$ is always an inner σ_j-derivation of A_j but the case A_j is a simple ring of characteristic $p_j\neq 0$ and $\sigma_j|_{Z(A_j)}=\operatorname{id}|_{Z(A_j)}$. In the later case $\delta_j(z_j^{p_j})=p_j z_j^{p_j-1}\delta_j(z_j)=0$, as

$z_j \in Z(A_j) = Z(A_j)^{\sigma_j}$. When δ_j is an inner σ_j-derivation, then $\delta_j(z_j) = 0$, as $z_j \in Z(A_j)^{\sigma_j}$. In this case we set $p_j = 1$. Then $\delta_j(z_j^{p_j}) = 0$, for any $1 \leq j \leq k$. Let $m = \prod_{j=1}^{k} p_j$. Then, putting together the above information, we get $\delta(z^m) = \delta_1(z_1^m) + \ldots + \delta_k(z_k^m) = 0$. This shows that the regular element z^m belongs to $Z^{\sigma,\delta}$ and proves that z is invertible in $R(Z^{\sigma,\delta})^{-1}$, i.e. Q is a localization of R with respect to regular elements from $Z^{\sigma,\delta}$.

(2). Let $1 = e_1 + \ldots + e_k$ be a decomposition of $1 \in Q$ into a sum of central primitive idempotents (i.e. $B_i = e_i R$, for all i).

Let us fix an element $a \in Z^{\sigma,\delta}$ which is regular in $Z^{\sigma,\delta}$ and $b \in Z$ such that $ab = 0$.

Assume that $b \neq 0$. Then, there exists an index s such that $ae_s = 0$. Eventually changing the numeration, we may assume that $s = 1$ and $\mathcal{A} = \{e_1, \ldots, e_l\}$ is the orbit of e_1 under the action of σ on the set $\{e_i \mid 1 \leq i \leq k\}$. Since $\sigma(a) = a$, we have $ac = 0$, where $c = \sum_{e_i \in \mathcal{A}} e_i$. Observe that the element c is (σ, δ)-invariant.

By the statement (1), $Q = R(Z^{\sigma,\delta})^{-1}$. Therefore, there exist an element $z \in Z^{\sigma,\delta}$ regular in R and $0 \neq u \in R$ such that $c = uz^{-1}$. Using the fact that the elements c and z are central (σ, δ)-invariant and z is regular in R, one can check that $u \in Z^{\sigma,\delta}$. Since a is regular in $Z^{\sigma,\delta}$ we obtain $0 \neq au = acz = 0$ This contradiction shows that $b = 0$ and this completes the proof of (2).

The statement (3) is a direct consequence of the fact that $Z(Q) = ZZ^{-1}$ and statements (1) and (2). □

Now we are in position to prove the main theorem of this section:

Theorem 3.7. *Suppose that R is a semiprime* PI *ring with the* ACC *on annihilators and σ is an injective endomorphism of R. Let Z denote the center of R and $Q = Q(R)$. The following conditions are equivalent:*

1. *$R[x; \sigma, \delta]$ is a* PI *ring.*
2. *$\mathrm{udim}_{Z^{\sigma,\delta}}(Z) = \mathrm{udim}_{Z(Q)^{\sigma,\delta}}(Z(Q))$ is finite.*
3. *The center of $R[x; \sigma, \delta]$ contains a nonconstant polynomial with a regular leading coefficient.*

If one of the above equivalent conditions holds, then every regular element from $Z^{\sigma,\delta}$ is regular in R, $Q = R(Z^{\sigma,\delta})^{-1}$ and $Z(Q)^{\sigma,\delta} = Z^{\sigma,\delta}(Z^{\sigma,\delta})^{-1}$.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. Then, by Proposition 1.6, $Q[x; \sigma, \delta]$ is a PI ring and Corollary 3.5 shows that $\mathrm{udim}_{Z(Q)^{\sigma,\delta}}(Z(Q)) < \infty$. The equality $\mathrm{udim}_{Z^{\sigma,\delta}}(Z) = \mathrm{udim}_{Z(Q)^{\sigma,\delta}}(Z(Q))$ is a direct consequence of Lemmas 3.6 and 1.7.

(2) $\Rightarrow$ (3). Suppose (2) holds. Then, by Corollary 3.5 applied to the ring Q, the ring $Q[x;\sigma,\delta]$ satisfies a polynomial identity and there exists a nonconstant polynomial f in the center of $Q[x;\sigma,\delta]$ with invertible leading coefficient. In particular, $R[x;\sigma,\delta]$ is also a PI ring and Lemma 3.6 shows that $Q = R(Z^{\sigma,\delta})^{-1}$. Hence, there exists an element $z \in Z^{\sigma,\delta}$ regular in R, such that $zf \in R[x;\sigma,\delta]$. Clearly zf is central in $R[x;\sigma,\delta]$ and has a regular leading coefficient.

(3) $\Rightarrow$ (1). Notice that $Z(R[x;\sigma,\delta]) \subseteq Z(Q[x;\sigma,\delta])$ and every regular element in R is invertible in Q. Thus, the statement (3) together with Corollary 3.5 show that the ring $Q[x;\sigma,\delta]$ satisfies a polynomial identity. This gives the thesis.

The above shows that conditions (1) $\div$ (3) are equivalent. The remaining statements from the theorem are direct consequences of Lemma 3.6. $\square$

Let us remark that the assumption in the above theorem, that the ring R satisfies the ACC condition on annihilators, is essential.

Example 3.8. Let $R = \prod_{i=1}^{\infty} \mathbb{C}$, where $\mathbb{C}$ denotes the field of complex numbers, and let σ be the automorphism of R which is the complex conjugation on every component $\mathbb{C}$ of R. Then $R[x;\sigma]$ is a PI ring, $R^\sigma = \prod_{i=1}^{\infty} \mathbb{R}$ and $\mathrm{udim}_{R^\sigma}(R)$ is infinite.

4 Noetherian Coefficient Ring

Throughout this section σ stands for an arbitrary, not necessarily injective, endomorphism of a ring R. $\mathcal{B}(R)$ denotes the prime radical of R.

The following result is due to Mushrub (Cf.[13]):

Lemma 4.1. *Suppose that R is a noetherian ring and $\ker\sigma \subseteq \mathcal{B}(R)$, then $\sigma(\mathcal{B}(R)) \subseteq \mathcal{B}(R)$.*

Proposition 4.2. *Suppose R is noetherian and $\ker\sigma \subseteq \mathcal{B}(R) = B$. Then $\sigma^{-1}(B) = B$ and σ induces an injective endomorphism $\bar{\sigma}$ of R/B.*

Proof. By Lemma 4.1, $\sigma(B) \subseteq B$. Thus σ induces an endomorphism $\bar{\sigma}$ of R/B.

By assumption, $\ker\sigma$ is a nilpotent ideal of R. Then, it is easy to check that also $\sigma^{-1}(I)$ is a nilpotent ideal, for any nilpotent ideal I of R. Thus, in particular, $\sigma^{-1}(B) \subseteq B$. Then $B \subseteq \sigma^{-1}(\sigma(B)) \subseteq \sigma^{-1}(B) \subseteq B$ and $\sigma^{-1}(B) = B$ follows. This, in turn, implies that $\bar{\sigma}$ is injective. $\square$

Proposition 4.3. *Suppose that σ is an endomorphism of a noetherian* PI *ring R such that* $\ker\sigma \subseteq B = \mathcal{B}(R)$. *Then $R[x;\sigma]$ is a* PI *ring if and only if the* *restriction of $\bar{\sigma}$ to the center Z of R/B is an automorphism of Z of finite order.*

Proof. By Proposition 4.2, σ induces an injective endomorphism $\bar{\sigma}$ of R/B. Since $\sigma(B) \subseteq B$ and B is nilpotent, $B[x;\sigma]$ is a nilpotent ideal of $R[x;\sigma]$. Therefore, as $(R/B)[x;\bar{\sigma}]$ is isomorphic to $R[x;\sigma]/B[x;\sigma]$, $R[x;\sigma]$ is a PI ring iff $(R/B)[x;\bar{\sigma}]$ is PI. Now the thesis is a direct consequence of Proposition 3.2. □

If σ is an automorphism of R, then $\sigma(\mathcal{B}(R)) = \mathcal{B}(R)$ and the above proposition is exactly Corollary 10[6], in this case.

Notice that, by Lemma 1.5(2), the ring $(R/B)[x;\bar{\sigma}]$ from the proof of the above theorem is semiprime. Therefore we have:

Remark 4.4. Suppose that σ is an endomorphism of a noetherian PI ring R such that $\ker\sigma \subseteq \mathcal{B}(R)$. Then $B(R[x;\sigma]) = \mathcal{B}(R)[x;\sigma]$.

It is known that if the ring R is not noetherian, then $B(R[x;\sigma])$ is not necessarily an extension of an ideal of the coefficient ring R even when σ is an automorphism of R. However, it was observed by Pascaud and Valette in [14] that $B(R[x;\sigma]) = \mathcal{B}(R)[x;\sigma]$, provided σ is an automorphism of R and the Ore extension $R[x;\sigma]$ satisfies a polynomial identity.

The following lemma is crucial in considering the case when $\ker\sigma$ is not included in the radical $\mathcal{B}(R)$ of R.

Lemma 4.5. *Let I be an ideal of R such that $\sigma(I) \subseteq I$. Suppose that R and $(R/I)[x;\sigma]$ are* PI *rings. Then $T = R[x;\sigma]/(I[x;\sigma]x)$ is also a* PI *ring.*

Proof. Since $\sigma(I) \subseteq I$, σ induces an endomorphism, also denoted by σ, on the factor ring R/I.

Notice that $R \cap I[x;\sigma]x = 0$, so we can consider R as a subring of T. Then I is also an ideal of T and T/I is isomorphic to $(R/I)[x;\sigma]$.

Let (x) denote the ideal of T generated by the natural image of x in T. Then $T/(x) \simeq R$. Therefore, as $I \cap (x) = 0$, there exists an embedding of T into $R \oplus (R/I)[x;\sigma]$ which, by assumption, is a PI ring. □

Notice that an endomorphism σ of R induces an endomorphism of the factor ring $R/\ker\sigma$. This endomorphism will also be denoted by σ. The above lemma gives us immediately:

Corollary 4.6. *Suppose that R is a* PI *ring. The following conditions are equivalent:*

1. *$R[x;\sigma]$ is a* PI *ring;*

2. *for any* $n \geq 0$, $(R/\ker\sigma^n)[x;\sigma]$ *is a* PI *ring;*

3. *there exists* $n \geq 0$ *such that* $(R/\ker\sigma^n)[x;\sigma]$ *is a* PI *ring.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1). Suppose $(R/\ker\sigma^n)[x;\sigma]$ is PI for some $n \geq 0$. We may assume $n \geq 1$. Let $I = \ker\sigma^n$. Then, by Lemma 4.5, the ring $T = R[x;\sigma]/(I[x;\sigma]x)$ is PI. Since $I = \ker\sigma^n$, $x^n I = 0$. Therefore $(I[x;\sigma]x)^{n+1} = 0$. This implies that $R[x;\sigma]$ satisfies a polynomial identity, □

When σ is an endomorphism, then $\{\ker\sigma^k\}_{k\geq 1}$ is an increasing sequence of ideals of R. Thus, when R is noetherian, there is $n \geq 1$ such that $\ker\sigma^n = \ker\sigma^m$ for any $m \geq n$ and σ induces an injective endomorphism, also denoted by σ, of the factor ring $R/\ker\sigma^n$.

Theorem 4.7. *Suppose* R *is a noetherian* PI *ring. Let* $n \geq 1$ *be such that* $\ker\sigma^n = \ker\sigma^{n+1}$ *and* $R' = R/I$*, where* $I = \ker\sigma^n$*. The following conditions are equivalent:*

1. $R[x;\sigma]$ *is a* PI *ring;*

2. $R'[x;\sigma]$ *is a* PI *ring;*

3. $\bar{\sigma}$ *is an automorphism of finite order on the center of* $R'/\mathcal{B}(R')$*.*

Proof. As we have seen in comments before the theorem, σ induces an injective endomorphism of R'. Thus, Proposition 4.3, shows that the statements (2) and (3) are equivalent.

The equivalence (1) $\Leftrightarrow$ (2) is given by Corollary 4.6. □

We have seen in Proposition 4.2 that if $\ker\sigma \subseteq \mathcal{B}(R)$ in a noetherian ring R, then $\sigma^{-1}(\mathcal{B}(R)) = \mathcal{B}(R)$. Hence $\ker\sigma^n \subseteq \mathcal{B}(R)$, for any $n \geq 1$. This means that $R'/\mathcal{B}(R') = R/\mathcal{B}(R)$ in the theorem above, i.e. Proposition 4.3 and Theorem 4.7 coincide when $\ker\sigma \subseteq \mathcal{B}(R)$.

Notice that, as the following standard example shows, the ring $R[x;\sigma]$ is not necessary noetherian even when it is PI and R is noetherian.

Example 4.8. Let K be a field and σ a K-linear endomorphism of the polynomial ring $R = K[y_1,\ldots,y_n]$ given by $\sigma(y_1) = 0$ and $\sigma(y_k) = y_{k-1}$ for $k > 1$. It is easy to see that $R[x;\sigma]$ is neither left nor right noetherian. By Theorem 4.7, $R[x;\sigma]$ satisfies a polynomial identity. In fact, since $\ker\sigma^n = (y_1,\ldots,y_n) = I$, $(R/I)[x;\sigma] = K[x]$ and the proof of Theorem 4.7 shows that $R[x;\sigma]$ satisfies the identity $[x_1,x_2]^{n+1} = 0$.

For any ring R, σ induces an injective endomorphism of R/I where $I = \sum_{k=1}^{\infty} \ker \sigma^k$. Thus one could hope that an analog of Theorem 4.7 could hold at least in the case I is a prime ideal of R (then, by Proposition 2.5, statements (2) and (3) are equivalent and (1) always implies (2)). However this is not the case as the following example shows.

Example 4.9. Let K be a field and σ a K-linear endomorphism of the polynomial ring $R = K[y_i \mid i \geq 1]$ given by $\sigma(y_1) = 0$ and $\sigma(y_k) = y_{k-1}$ for $k > 1$. Then $I = \sum_{k=1}^{\infty} \ker \sigma^k = (y_1, y_2, \ldots)$ and $R[x;\sigma]/I[x;\sigma] \simeq R/I[x;\sigma] \simeq K[x]$ is a PI ring.

We claim that $R[x;\sigma]$ does not satisfy a polynomial identity by showing that, for any $m \geq 2$ and $k \geq 1$, $R[x;\sigma]$ does not satisfy the identity $S_m(x_1, \ldots, x_m)^k$, where $S_m(x_1, \ldots, x_m)$ denotes the standard identity in indeterminates $x_1, \ldots, x_m$. To this end, let us fix $n = n(m,k)$ such that $n > mk$. Then $S = S_m(y_n t, \ldots, y_{n+m-1} t) = (y_n^m + f)t^m$ for some suitable $f \in R[x;\sigma]$ such that $\deg_{y_n}(f) < m$. The choice of n implies that $y_n^m + f \notin \ker \sigma^{(k-1)m}$. Hence $S^k \neq 0$ follows, as R is a domain.

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